D-brane superpotentials and RG flows on the quintic

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Abstract: The behaviour of D2-branes on the quintic under complex structure deformations is analysed by combining Landau-Ginzburg techniques with methods from conformal field theory. It is shown that the boundary renormalisation group flow induced by the bulk deformations is realised as a gradient flow of the effective space time superpotential which is calculated explicitly to all orders in the boundary coupling constant.

Keywords: D-branes, Calabi-Yau manifolds, Matrix factorisations.

Contents

1. Introduction

A good understanding of the moduli spaces of string backgrounds is a central issue in string theory. In many situations of interest the backgrounds involve D-branes, and then there are two kinds of moduli: D-brane moduli (that describe the position *etc.* of the branes), and closed string moduli (that parametrise the different closed string backgrounds). Obviously, these moduli are not independent of one another, and in particular the D-brane moduli space depends on the specific closed string background that is being considered. It is an important problem (for example in the context of stabilising all moduli) to understand this dependence in more detail.

In this paper we study this problem for a certain class of supersymmetric D-branes on the quintic Calabi-Yau threefold. The branes we consider wrap rational curves considered in [1], and extend along the uncompactified dimensions. We will be interested only in the internal geometry, and hence refer to these branes as D2-branes rather than D5-branes. The 2-cycles that are being wrapped can be specified by linear equations, and at the Gepner point of the quintic they actually form a complex 1-dimensional moduli space. This moduli space can be described explicitly in terms of matrix factorisations of the associated Landau-Ginzburg model [2, 3].

As we perturb the closed string theory by a complex structure deformation, only finitely many D2-branes remain supersymmetric. In fact, the number of holomorphic spheres has been counted, using mirror symmetry, in [4]. A generic D2-brane will therefore have to adjust itself to the new background. This can be understood from a world-sheet point of view following [5]:* the bulk deformation induces a boundary renormalisation group (RG) flow that drives the brane to one of the allowed discrete configurations. More specifically, the resulting flow is determined by certain bulk-boundary OPE coefficients. These coefficients are part of the topological sector of the theory and can thus be calculated using standard Landau-Ginzburg techniques following [8]; this allows us to describe the RG flow completely. As we shall prove fairly generically, the resulting flow is the gradient flow of the space-time effective superpotential.

We can also interpret our calculation as a method to determine certain contributions of the effective space-time superpotential exactly. Since the matrix factorisation description applies to the full moduli space, we can use it to calculate the open-closed disc correlation functions with one boundary and one bulk insertion *at every point* in the boundary moduli space. In particular, we can take the boundary field to correspond to the modulus of the brane moduli space, and then the correlator can be interpreted as the derivative (with respect to the boundary modulus) of the full generating function of the bulk one-point function with an arbitrary number of boundary fields. By integration we can thus determine from it the exact contribution to the space-time effective superpotential that is linear in the bulk modulus. The integral can be performed explicitly, and we find that the resulting superpotential is a linear combination of certain hypergeometric functions. Partial results for these contributions to the effective spacetime superpotential have been obtained before in [9], and using a complementary approach more recently in [10].

Geometrically, the resulting formula has the form of an integral of a holomorphic one-form along a path on a Riemann surface parametrising the possible brane positions. It is therefore very reminiscent of the analysis of [11] (see also [12]), who expressed the superpotential for branes wrapping curves in terms of the Abel Jacobi map by evaluating the holomorphic Chern Simons action.

Various methods to calculate the effective superpotential have been used before in the literature. From a geometric point of view one can calculate the superpotential as the action of a holomorphic Chern Simons theory [13]. For branes wrapping 2-cycles this

*A similar effect has been studied from an open string field theoretic point of view in [6, 7].

simplifies to the integral over certain 3-chains. For compact Calabi-Yau manifolds (such as the quintic), such integrals are however difficult to calculate.

Another approach is to make use of conformal field theory techniques to evaluate the correlators in perturbation theory. This technique was successful in the case of minimal models where all correlators could be calculated by making use of the consistency conditions imposed by the sewing relations [14]. For the case of Calabi-Yau manifolds, however, these consistency conditions by themselves do not suffice to determine the superpotential completely. On the other hand, explicit conformal field theory calculations of correlators are difficult and can at present only be evaluated at rational points (and to low orders in perturbation theory).

For the specific case of the quintic, there are also some explicit results in the literature. For example, for the Lagrangian (A-type) branes whose description in terms of conformal field theory was found in [9], the first few terms of the superpotential have been determined in [15], using rational conformal field theory techniques following [16]. The same problem has also been addressed using Landau-Ginzburg techniques in [10, 17]. More recently, the exact superpotential has been calculated on the mirror B-side in [18] by guessing the correct Picard-Fuchs equation; this has also been checked against the instanton expansion of the A-side that can be interpreted in terms of the counting of holomorphic discs. Finally, superpotential terms for B-type D-branes described as pull-backs of sheaves from the embedding $\mathbb{P}_4$ have been calculated in [19] using linear sigma-model techniques. It was shown there that the superpotentials obtained in this way capture the geometry and obstruction theory of the corresponding bundles and sheaves.

The paper is organised as follows. In the remainder of this section we briefly review the geometric description of the quintic and the relevant family of D2-branes. Section 2 provides background information about matrix factorisations and their connection to Gepner models. Moreover the above family of branes is described from a matrix factorisation point of view. In section 3 we study the RG flow that is induced by certain complex structure deformations of the Gepner point. In particular, we find that the RG flow is a gradient flow and we determine the relevant potential explicitly. In section 4 we explain that the potential can in fact be identified with a certain contribution to the space-time effective superpotential. Section 5 contains our conclusions. There are two appendices to which some of technical calculations have been deferred.

1.1 The model

In this paper we consider D2-branes wrapping holomorphic 2-cycles of the quintic. Our starting point is the Fermat quintic given by the following hypersurface in $\mathbb{P}_4$

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0 \subset \mathbb{P}_4 \ . \tag{1.1}$$

We are interested in a special family of branes wrapping rational curves, which has been studied from a mathematical point of view in [1] and from a physics point of view in [10], see [9] for earlier work. More concretely, the family of curves we have in mind is given by

$$(x_1, x_2, x_3, x_4, x_5) = (u, \eta u, av, bv, cv) \ , \quad \text{where} \quad a^5 + b^5 + c^5 = 0 \ . \tag{1.2}$$

Here $a, b, c \in \mathbb{C}$, η is a 5^{th} root of -1, and (u, v) parametrise a $\mathbb{P}_1$. The three complex parameters a, b, c are subject to projective equivalence and the complex equation in (1.2), so that the above equations describe a one parameter family of $\mathbb{P}_1$'s. In fact there are 50 such families since there are 10 possibilities to pick a pair of coordinates that are proportional to u, and 5 choices for η. These families intersect along the lines

$$x_i - \eta x_j = 0 \ , \quad x_k - \eta' x_l = 0 \ , \quad x_m = 0 \ , \tag{1.3}$$

where i, j, k, l, m are all disjoint and η and η' are 5^{th} roots of -1. For example, the set

$$x_1 - \eta x_2 = 0 \ , \quad x_3 - \eta' x_4 = 0 \ , \quad x_5 = 0 \tag{1.4}$$

describes a particular $\mathbb{P}_1$ in (1.2) with $c = 0$, $a = \eta'$, $b = 1$. Likewise, it describes a $\mathbb{P}_1$ in the family

$$(x_1, x_2, x_3, x_4, x_5) = (av, bv, u, \eta' u, cv) \tag{1.5}$$

with $a = 1$, $b = \eta$ and $c = 0$. Starting from such a configuration, one can thus move along either of the two families of which this $\mathbb{P}_1$ is part. However, once one has started to move away in one direction, the other becomes obstructed [1]. For concreteness we shall mainly consider in the following the family of curves associated to (1.2) although everything we say can be easily generalised to the other classes of branes.

From a conformal field theory point of view, the existence of the above families of $\mathbb{P}_1$'s implies that the open string spectrum of every corresponding brane contains an exactly marginal boundary operator which we shall denote by ψ_1. At the above intersection points there will be a second exactly marginal operator which we shall call ψ_2 [10]. The fact that moving away in one direction obstructs the other should imply that the effective superpotential contains a term of the form

$$\mathcal{W}(\psi_1, \psi_2) = \psi_1^3 \psi_2^3 \ . \tag{1.6}$$

This was argued on physical grounds in [9] and later confirmed in [10]. Recently it was shown in [20] that (1.6) is already the full superpotential for the fields ψ_1 and ψ_2. We shall reproduce this result, using somewhat different methods, at the end of section 2.

The above discussion applies to the Gepner point of the quintic, where the hypersurface is described by equation (1.1). It is well known that at a generic point in the complex structure moduli space of the quintic, there are only discretely many (2875) rational 2-cycles; in particular there are therefore no continuous families of $\mathbb{P}_1$'s if we perturb the theory away from the Gepner point. Geometrically, this means that at a generic point in the above moduli space of branes, the complex structure deformations are obstructed, as has already been discussed in [1]. From a world-sheet point of view this should therefore mean that the effective superpotential contains a term of the form

$$\mathcal{W}(\psi_1, \psi_2, \Phi_i) = \psi_1^3 \psi_2^3 + \sum_i \Phi_i \, F_i(\psi_1, \psi_2) + \cdots \ , \tag{1.7}$$

where the Φ_i describe the different complex structure deformations.

In this paper we shall mainly consider the special deformations of the quintic described by

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_1^3 s^{(2)}(x_3, x_4, x_5) = 0 \ , \tag{1.8}$$

where $s^{(2)}$ is a polynomial of degree 2 in x_3, x_4 and x_5, The only curves that survive this deformation are those for which

$$a^5 + b^5 + c^5 = 0 \qquad \text{and} \qquad s^{(2)}(a, b, c) = 0 \ . \tag{1.9}$$

These equations determine a discrete set of points; in fact, counting multiplicities there are precisely 10 solutions, as follows from Bezout's theorem.

The deformations (1.8) are special in that the term linear in Φ in (1.7) is independent of ψ_2. In this case we can then determine the function $F(\psi_1, \psi_2)$ exactly, and thus give a complete description for how the system behaves under the corresponding bulk perturbation; this will be described in detail in section 3. As we shall see, the bulk perturbation induces a boundary RG flow that is the gradient flow of the function F; in particular the solutions to (1.9) are precisely the critical points of F.

2. Branes on the quintic

Let us begin by constructing the family of D2-branes (1.2) at the Fermat point in the Landau-Ginzburg model description.

2.1 The matrix factorisations description

At the Fermat point the quintic is described by the Gepner model corresponding to five copies of the $N = 2$ minimal model at $k = 3$ [21, 22, 23, 24]. The branes of interest are B-type branes of this superconformal field theory. As we shall see, isolated D-branes can be constructed as permutation branes in conformal field theory [25] (see also [26]), but in order to understand the full moduli space of branes we need different methods. We shall make use of the fact that in the associated Landau-Ginzburg (LG) theory, B-type branes can be described as matrix factorisations of the LG superpotential W [2, 3, 8, 27, 28, 29] (following unpublished work of Kontsevich). This formulation is particularly well adapted to the investigation of brane moduli spaces, as has been observed in various contexts [30, 10, 31, 32, 33].

At the Gepner point the relevant LG superpotential is

$$W_0 = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \ . \tag{2.1}$$

A matrix factorisation of W_0 is an operator Q that squares to the LG superpotential

$$Q^2 = W_0 \mathbb{1} \ . \tag{2.2}$$

Q is the boundary part of the BRST operator, and together with the bulk BRST charge squares to 0. In particular, Q is fermionic and can be expressed as a linear combination of

(non-BRST closed) fermionic operators π^i and their conjugates $\bar{\pi}^i$, $i = 1, \ldots, n$, that live at the boundary,

$$Q = \sum_{i=1}^{n} \left(\pi^i J_i + \bar{\pi}^i E_i \right) . \tag{2.3}$$

These fermions form a 2^n dimensional representation of the Clifford algebra

$$\{\pi^i, \bar{\pi}^j\} = \delta^{ij} \ , \qquad \{\pi^i, \pi^j\} = \{\bar{\pi}^i, \bar{\pi}^j\} = 0 \ . \tag{2.4}$$

The square of Q is given by

$$Q^2 = \Big(\sum_i E_i J_i \Big) \cdot \mathbb{1} \tag{2.5}$$

and hence Q defines a matrix factorisation if

$$W = \sum_i E_i J_i \ . \tag{2.6}$$

Turning the argument around, whenever W can be written in the form (2.6) a suitable matrix factorisation is given by (2.3). The matrix factorisation description captures all topological aspects of the corresponding D-branes. For example, one can determine from Q the topological part of the open string spectrum and the topological RR charges, *etc.* What will be most important for our purposes is the Kapustin-Li formula [8] that allows one to calculate bulk-boundary correlators (or boundery three point functions) exactly. If we denote a topological bulk field by Φ and the boundary field by ψ, then the disc correlator is

$$\langle \Phi \, \psi \rangle = \mathrm{Res} \ \Phi \frac{\mathrm{STr} \left[\, \partial_{x_1} Q \ldots \partial_{x_5} Q \, \psi \right]}{\partial_{x_1} W \ldots \partial_{x_5} W} \ , \tag{2.7}$$

where the residue is taken at the critical points of LG superpotential W. More details about this formula can be found in [8, 29].

Strictly speaking, to find an LG description of the quintic one has to consider an orbifold of the theory (2.1). This $\mathbb{Z}_5$ orbifold projects onto states with integer $U(1)$ charge in the closed string sector. As usual, the consequence for the open string sector is [41, 17, 42] that we need to specify in addition a representation of the orbifold group on the Chan-Paton labels. The open string spectrum is then given by the $\mathbb{Z}_5$ invariant part of the cohomology of the BRST operator. In the following, the additional representation label will play no further role, since we will only consider a single D-brane with an arbitrary but fixed representation label.

The D2-branes of interest correspond to a family of matrix factorisations that can be constructed as follows, using ideas similar to what was done in [32, 34] (see also [17]). We define

$$J_1 = x_1 - \eta x_2 \ , \qquad J_4 = a x_4 - b x_3 \ , \qquad J_5 = c x_3 - a x_5 \ , \tag{2.8}$$

and look for common solutions of $J_1 = J_4 = J_5 = 0$ and $W = 0$. If η is a fifth root of -1 and $a \neq 0$, we get a solution if

$$a^5 + b^5 + c^5 = 0 \ . \tag{2.9}$$

If this is the case we can use the Nullstellensatz to write

$$W_0 = J_1 \cdot E_1 + J_4 \cdot E_4 + J_5 \cdot E_5 \ , \tag{2.10}$$

where E_i are polynomials in x_j. We then obtain a matrix factorisation by the procedure outlined above. More specifically, we introduce 8×8 matrices π^i and $\bar{\pi}^i$, $i = 1, 4, 5$, that form a representation of the Clifford algebra, and obtain a family of matrix factorisations $Q(a, b, c)$

$$Q(a, b, c) = \sum_{j=1,4,5} \left(J_j \pi^j + E_j \bar{\pi}^j \right) \ . \tag{2.11}$$

By construction $Q(a, b, c)$ satisfies then $Q(a, b, c)^2 = W_0 \cdot \mathbb{1}$.

Following the geometrical interpretation of matrix factorisations elaborated in [35, 36, 37] (see also [38]) these matrix factorisations provide the LG-description of the D2-branes described in section 1.1. Indeed, read as equations in $\mathbb{P}_4$, the equations $J_1 = J_3 = J_5 = 0$ describe precisely the geometrical lines (1.2).

The moduli space of such branes has complex dimension one. Indeed, it is straightforward to see that rescaling (a, b, c) by a common factor results in an equivalent factorisation; thus (2.9) can be thought of as an equation in $\mathbb{CP}^2$, and hence describes a one-complex-dimensional curve.† Furthermore we note that special points on this curve correspond to standard permutation branes [25]: for example for $a \neq 0$ and $b = 0$ we may use the projective equivalence to set $a = 1$. Then c must be a fifth root of -1, leading precisely to a permutation factorisation of the form discussed in [39, 40]. This identification is also in agreement with the analysis of [41, 39] where it was shown that one of these matrix factorisations carries indeed the charge of a D2-brane.

2.2 The fermionic spectrum

The fact that these matrix factorisations form a 1-complex dimensional moduli space means that at every point in the moduli space the open string cohomology contains at least one fermion of $U(1)$-charge one. Indeed, this is just the matrix factorisation analogue of the fact that each such D-brane must have an exactly marginal boundary operator in its spectrum. From a matrix factorisation point of view, the corresponding fermion can be easily constructed. Since by assumption $a \neq 0$, we may always rescale the parameters so that $a = 1$. Let us first consider a generic point in moduli space where $bc \neq 0$. We then have a family of factorisations parametrised by (b, c) subject to $b^5 + c^5 = -1$. As long as $c \neq 0$, we can locally solve this equation for c, *i.e.* we can express $c \equiv c(b)$, and thus obtain a matrix factorisation $Q(b)$. Since W_0 does not depend on (a, b, c), it then follows that

$$\{Q(b), \partial_b Q(b)\} = 0 \tag{2.12}$$

which is precisely the condition for $\psi = \partial_b Q(b)$ to define a fermion of the cohomology defined by $Q(b)$. For the case under consideration, we find explicitly

$$\psi_b \equiv \partial_b Q(b) = -x_3 \, \pi^4 - \frac{b^4}{c^4} x_3 \, \pi^5 + (\partial_b E_4) \, \bar{\pi}^4 + (\partial_b E_5) \, \bar{\pi}^5 \ , \tag{2.13}$$

†In the above description we have not treated the three variables a, b and c on an equal footing, and hence a could not be zero. It should be clear, however, that we can also use a different chart in which $a = 0$ is possible. In this way we can obtain a matrix factorisation associated to (a, b, c) provided that not all three a, b and c are simultaneously zero and that (2.9) holds. See [17] for an explicit change of coordinates in a different example.

where we have used that

$$\left.\frac{\partial b}{\partial c}\right|_a = -\frac{c^4}{b^4} \ . \tag{2.14}$$

One can show by explicit computation (see appendix A) that ψ_b is non-trivial in cohomology. Obviously, we could have equally expressed $b \equiv b(c)$ (for $b \neq 0$) and written $Q(a,b,c) \equiv Q(c)$. Then the derivation with respect to c also defines a fermion

$$\psi_c \equiv \partial_c Q(c) = \frac{c^4}{b^4} x_3\, \pi^4 + x_3\, \pi^5 + (\partial_c E_4)\, \bar{\pi}^4 + (\partial_c E_5)\, \bar{\pi}^5 \ . \tag{2.15}$$

It is easy to see that for $bc \neq 0$ so that both ψ_b and ψ_c are well defined, $\psi_b \cong \psi_c$ in cohomology. In the following we shall denote the equivalence class to which ψ_b and ψ_c belong by ψ_1. More specifically, we shall usually take $\psi_1 \equiv \psi_b$ and assume that $c \neq 0$.

The full fermionic cohomology of $Q(a,b,c)$ at $U(1)$-charge 1 is however bigger: in addition to ψ_1 it also contains a second fermion that we shall call ψ_2. This is explained in appendix A, where ψ_2 is explicitly constructed (for $c \neq 0$). In general, however, ψ_2 does not define a modulus. In fact, using the Kapustin-Li formula [8] one easily finds that

$$B_{\psi_2\psi_2\psi_2} = -\frac{2}{5}\eta^4 \frac{b^3}{c^9} \ . \tag{2.16}$$

Unless $b = 0$ the three-point function of ψ_2 on the boundary does not vanish, and hence ψ_2 is not an exactly marginal boundary field [43]. This shows that at generic points in the moduli space (2.9) there is only one exactly marginal operator, whereas at the special point $b = 0$ an additional marginal operator appears, indicating an additional branch of the moduli space. This is in nice agreement with the geometric analysis of section 1.1 since at $b = 0$ the above moduli space intersects with the branch where the roles of J_1 and J_5 can be interchanged. In fact, this can also be seen from the explicit formula for ψ_2, see (A.12) in appendix A.

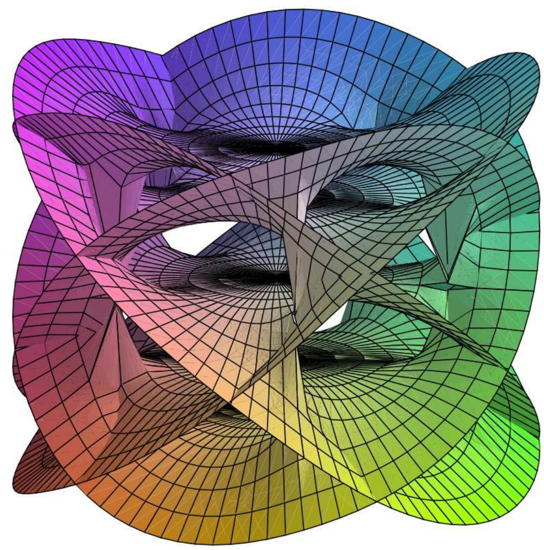

Figure 1: Riemann surface associated to the D-brane moduli space, consisting of five copies of the complex plane. The real and imaginary part of b have been plotted horizontally, the vertical axis is the imaginary part of c. The five sheets arranged vertically at $b = 0$ reflect the five possibilities for $c^5 = -1$.

The three-point function (2.16) verifies the superpotential term (1.6) that was already obtained in [10] by other means. Furthermore, after rescaling $\psi_2 \mapsto \hat{\psi}_2 = c^3\psi_2$, the b-dependence of the three-point function for $\hat{\psi}_2$ is simply proportional to b^3. (Recall that $c \equiv c(b)$.) Using the arguments of section 4.1 this then implies that, with respect to this normalisation, the effective superpotential does not contain any higher order contributions (in ψ_1) to the term $\psi_1^3\,\hat{\psi}_2^3$ in (1.6). This is in agreement with the recent analysis of [20].

3. Bulk perturbation

Now we want to consider the bulk perturbation of the above Gepner model by the bulk operator Φ, *i.e.* we consider the perturbed superpotential

$$W = W_0 + \lambda\,\Phi\ , \qquad \Phi = x_1^3\, s^{(2)}(x_3, x_4, x_5)\ , \tag{3.1}$$

where $s^{(2)}$ is the polynomial of section 1.1 that we expand as[‡]

$$s^{(2)}(x_3, x_4, x_5) = \sum_{q+r+s=2} s^{(2)}_{qrs}\, x_3^q\, x_4^r\, x_5^s\ . \tag{3.2}$$

From a conformal field theory point of view the perturbation is generated by an exactly marginal bulk field in the *cc* ring. We want to understand what happens to the D-branes described by the moduli space (2.9) under this perturbation. We shall be able to give a fairly complete description of this problem by combining the ideas of [5] with matrix factorisation techniques. In particular, this will allow us to calculate the effective superpotential for the boundary parameters (a, b, c) exactly.

One way to address this problem is to study the deformation theory of matrix factorisations, following [17] (see also [10]). Suppose that Q_0 is a factorisation of W_0. Then we ask whether we can find a deformation Q of Q_0, *i.e.*

$$Q = Q_0 + \lambda Q_1 + \lambda^2 Q_2 + \cdots \tag{3.3}$$

such that $Q^2 = W_0 + \lambda\Phi$. Expanding this equation to first order in λ, we find the necessary condition that Φ must be exact with respect to Q_0, *i.e.* of the form $\Phi = \{Q_0, \chi\}$ for some χ. In general this condition will not be met; for example for the case at hand where $Q_0 \equiv Q(a, b, c)$ and Φ is given by (3.1), we find that Φ is exact if and only if

$$a^5 + b^5 + c^5 = 0 \qquad \text{and} \qquad s^{(2)}(a, b, c) = 0\ . \tag{3.4}$$

On the other hand, if this condition *is met*, it is easy to see that we *can* in fact extend the matrix factorisation for arbitrary (finite) values of λ. Indeed, if we consider the same ansatz as in (2.8), it is clear that we can find a joint solution to $J_1 = J_4 = J_5 = 0$ and $W = W_0 + \lambda\Phi = 0$ if (a, b, c) satisfies (3.4). It then follows by the same arguments as above that there exists a matrix factorisation for all values of λ (that is by construction a deformation of $Q(a, b, c)$).

Unless $s^{(2)} \equiv 0$, the set of constraints (3.4) has only finitely many discrete solutions; in fact, counting multiplicities, there are precisely 10 solutions, as follows from Bezout's theorem. This ties in nicely with our geometric expectations since at a generic point in the complex structure moduli space only finitely many holomorphic 2-cycles exist.

[‡]Everything we are going to say is essentially unchanged if we were to replace x_1^3 by an arbitrary third order polynomial in x_1 and x_2.

3.1 Combining with conformal field theory

As we have just seen, for $\lambda = 0$ we have a one-parameter family of superconformal D2-branes, while for $\lambda \neq 0$ only discrete possibilities remain. The situation is therefore very analogeous to the example studied in [5]. There a general conformal field theory analysis of this problem was suggested that we now want to apply to the case at hand.

In [5] the coupled bulk and boundary deformations of a boundary conformal field theory were studied, and the resulting renormalisation group identities were derived. It was found that an exactly marginal bulk operator may cease to be exactly marginal in the presence of a boundary. If this is the case it will induce a renormalisation group flow on the boundary that will drive the boundary condition to one that is again conformal with respect to the deformed bulk theory. If we denote the boundary coupling constant corresponding to the boundary field ψ_j of conformal weight h_j by μ_j, then the perturbation by the exactly marginal bulk operator $\lambda\Phi$ will induce the RG equation

$$\dot{\mu}_j = (1 - h_j)\mu_j + \frac{\lambda}{2} B_{\Phi\psi_j} + \mathcal{O}(\mu\lambda, \lambda^2, \mu^2) \ , \tag{3.5}$$

where $B_{\Phi\psi_j}$ is the bulk-boundary operator product coefficient. Since the first term in (3.5) damps the flow of any irrelevant operators, it is sufficient to study this equation only for the marginal or relevant boundary fields, *i.e.* for those that satisfy $h_j \leq 1$.

For the case at hand, we do not have an explicit conformal field theory description of the D-branes away from the specific points where $abc = 0$. On the other hand, we know (based on supersymmetry) that the open string spectrum will not contain any relevant (tachyonic) operators. Furthermore, the above discussion suggests that everywhere in moduli space each brane has precisely two marginal operators in its spectrum, namely the operators corresponding to the open string fermions described by ψ_1 and ψ_2 — see appendix A for details. The two boundary operators ψ_1 and ψ_2 are topological, and so is the bulk perturbation Φ. In particular, this implies that we can determine the coefficients $B_{\Phi\psi_1}$ and $B_{\Phi\psi_2}$ that are important for the RG equations using *topological methods*, without having to solve the full conformal field theory (which would be impossibly difficult)!

Using the Kapustin-Li formula (2.7) we find (we are working in a patch where $a = 1$)

$$B_{\Phi\psi_2} = 0 \tag{3.6}$$

for all (a, b, c), as well as

$$B_{\Phi\psi_b} = \frac{\eta^4}{25} c^{-4} s^{(2)}(1, b, c) \ , \tag{3.7}$$

and similarly for

$$B_{\Phi\psi_c} = -\frac{\eta^4}{25} b^{-4} s^{(2)}(1, b, c) \ . \tag{3.8}$$

All of these calculations were performed in the unperturbed bulk theory. Since the bulk-boundary coupling between Φ and ψ_2 vanishes (3.6), this field is not switched on by Φ. The RG flow will therefore only involve ψ_1, and for this we find

$$\dot{b} = \lambda \frac{\eta^4}{50} c^{-4} s^{(2)}(1, b, c) \ , \tag{3.9}$$

or

$$\dot{c} = -\lambda \frac{\eta^4}{50} b^{-4} s^{(2)}(1,b,c) \ . \tag{3.10}$$

In particular, we see that the solutions to (3.4) are precisely the fixed points under the RG equation. Thus any brane described by (a,b,c), will flow to one of these 10 fixed points under the RG flow.

3.2 RG flow as gradient flow

Actually, the above RG flow is a gradient flow, as was also the case in the example studied in [5].[§] In fact, we can integrate the RG equation for b in (3.9) to

$$\dot{b} = \partial_b \mathcal{W}(a,b,c) \ , \tag{3.11}$$

where $\mathcal{W}(a,b,c)$ is evaluated on the moduli space (2.9) with $a^5+b^5+c^5=0$ and we have rescaled $a=1$. Similarly, the same function $\mathcal{W}(a,b,c)$ also controls the RG equation for c in (3.10)

$$\dot{c} = \partial_c \mathcal{W}(a,b,c) \ , \tag{3.12}$$

where again $a=1$ and we regard b as a function of c via the constraint $a^5+b^5+c^5=0$. To determine $\mathcal{W}(a,b,c)$ explicitly we need to integrate

$$\int_{b_0}^{b} db' B_{\Phi\psi_b} = \frac{\eta^5}{25} \int_{b_0}^{b} db' c^{-4} s^{(2)}(a,b',c) \ . \tag{3.13}$$

The integral is along a line on the Riemann surface starting at a fixed reference point b_0 that we take to be 0 and ending at b. Since b parametrises the brane moduli space, it has a natural physical interpretation as the position of the brane. The integrand is a holomorphic one-form on the Riemann surface parametrising the moduli space, see appendix B for more details. The potential therefore has a natural geometric interpretation as the Abel-Jacobi map associated to a one-form on the Riemann surface whose points label the brane positions. Which particular one-form is to be integrated is determined by the bulk deformation under consideration.

Since the integrals of such forms are known, we can give explicit formulae for $\mathcal{W}(a,b,c)$ in each patch. As explained in appendix B, in the patch where $a=1$ and $c \neq 0$ (so that $c \equiv c(b)$ is well defined) one obtains

$$\mathcal{W}(1,b,c) = \lambda \frac{\eta^4}{50} \sum_{q+r+s=2} \frac{1}{r+1} s^{(2)}_{qrs} (-b)^{r+1} \, {}_2\mathrm{F}_1(\tfrac{r+1}{5}, 1-\tfrac{s+1}{5}; 1+\tfrac{r+1}{5}; -b^5) \ . \tag{3.14}$$

It is also checked there that this function satisfies both (3.11) and (3.12).

By combining abstract conformal field theory arguments with topological methods we can thus give a complete description of the RG flow: the D2-brane simply follows the gradient flow of $\mathcal{W}$ to arrive at one of its local minima, which are precisely the points characterised by (3.4). As in [5], in the RG scheme in which we always remain in the

[§]For exactly marginal bulk deformations this may in fact follow from the analysis of [44].

original moduli space, this analysis is exact in the boundary moduli, and first order in the bulk coupling constant. Obviously the picture we have found ties in very nicely with the geometric expectations of section 1.1.

We should note that it is crucial in this analysis that the bulk perturbation by Φ does not switch on ψ_2, *i.e.* that $B_{\Phi\psi_2} = 0$. Otherwise the bulk perturbation would switch on a boundary field that would lead us out of the original moduli space and we would not be able to iterate the RG equations. This is the reason why we restricted our analysis to the bulk perturbations of the form described in (3.2).

4. Superpotentials

The function $\mathcal{W}$ has actually an interpretation in terms of the effective spacetime superpotential, as we shall now explain.

4.1 Gradient flow of the superpotential

In the above we have seen explicitly that the RG flow is a gradient flow of a potential. This potential is precisely the contribution to the effective superpotential $\mathcal{W}$ that is first order in the bulk field Φ and exact in the boundary field ψ_1. To see this we simply note that the term that appears on the right hand side of (3.5) is the bulk-boundary coefficient that involves one insertion of the bulk field Φ and one insertion of the boundary field ψ_1 (that couples to μ). This bulk-boundary correlator was evaluated at an *arbitrary* point in the brane moduli space; if we start around any given point of the brane moduli space, the above expression therefore involves an arbitrary number of insertions of ψ_1 (that allow one to move around this brane moduli space). Thus the right-hand-side of (3.5) is the generating function describing symmetrised correlators involving an arbitrary number of boundary fields ψ_1, together with one insertion of the boundary field ψ_1 and one insertion of the bulk field Φ. We can produce the insertion of the boundary field ψ_1 by taking a derivative with respect to the corresponding boundary coupling constant. It thus follows that the function $\mathcal{W}$ (that we obtained by integrating up the right hand side of (3.5)) is precisely the generating function of one bulk field Φ with an arbitrary number of boundary fields. It therefore defines the corresponding contribution of the effective superpotential.

It is also clear from this argument that this method can be applied to calculate the corresponding terms of the effective superpotential for an arbitrary bulk deformation, not just one of the form (3.2). For the other cases, the result is however trivial: the complex structure deformations (3.2) are the only monomials (instead of x_1^3 we may also allow for an arbitrary third order polynomial in x_1 and x_2) for which the bulk-boundary OPE coefficient with ψ_1 is non-zero. Thus to first order in the bulk perturbation the above terms are the only terms that appear in the effective superpotential. It should also be obvious how to perform the same analysis for the other (45) families of D2-branes.

4.2 Interpretation in terms of Chern-Simons theory

The observation that superpotentials can be viewed as Abel-Jacobi maps for certain one-forms on Riemann surfaces labelling the brane positions have appeared before in [11, 45]

(see also [12]). In their approach, these superpotentials were calculated using geometric methods, which yields the exact result on the B-side. Namely, the superpotential is given by the action of a holomorphic Chern Simons theory living on the brane [13] which for branes wrapping holomorphic curves in the internal dimensions can be reduced to an integral of the holomorphic 3-form over a 3-chain. These integrals can be explicitly evaluated in the examples studied in [11, 45] and are effectively reduced to one-dimensional integrals along a path with two end-points on the Riemann surface. These one-dimensional integrals give the superpotential in terms of the Abel Jacobi map for the one-form, similar to our case.

In later developments, [46, 47, 48, 49] the 3-chain integrals mentioned above appeared in certain differential equations extending the well-known Picard Fuchs equations governing the closed string sector. In the context of the quintic, such a differential equation has been proposed for a different D-brane in [18], resulting in a full instanton expansion on the mirror A-side (see [34, 50] for a result on the torus). In our example, the appearance of hypergeometric functions as effective potentials suggests to consider the hypergeometric differential equation of which these are solutions. We speculate that this differential equation is part of the Picard Fuchs equations governing the open string sector in the sense of [46, 47, 48].

4.3 Integrability

The fact that the bulk-boundary correlator could be integrated up to a generating function of the boundary fields ψ_a with the insertion of the given bulk field Φ is something we can show in generality. If we denote the relevant bulk-boundary correlator by $B_{\Phi\psi_a}$, then the integrability condition is simply that

$$\partial_{a_1} B_{\Phi\psi_{a_2}} = \partial_{a_2} B_{\Phi\psi_{a_1}} \ . \tag{4.1}$$

Similar relations are well-known to hold in the bulk [51]; for boundary correlators there are usually ordering ambiguities for the operators on the boundary, and hence similar relations can only be expected to hold for the symmetrised correlation functions. However, the disc correlation functions are still cyclically symmetric in the boundary insertions and totally symmetric in the bulk insertions [14] from which one can infer that (4.1) holds. The following is essentially a review of the argument of [14], which again is similar to [51]. We will work in the topologically twisted theory, where G^+ becomes the BRST current Q, while G^- becomes an $h = 2$ field of the topological theory. The starting point are the Ward identities for the open-closed correlators [14]

$$\begin{aligned}
0 = \ & \langle \oint \xi(w) G(w) \psi_{a_1}(\tau_1) \cdots \psi_{a_m}(\tau_m) \Phi_{i_1}(z_1,\bar z_1) \cdots \Phi_{i_s}(z_s,\bar z_s)\rangle \\
= \ & \sum_{k=1}^{m} \pm \xi(\tau_k)\, \langle \psi_{a_1}(\tau_1) \cdots \psi^{(1)}_{a_k}(\tau_k) \cdots \psi_{a_m}(\tau_m)\, \Phi_{i_1}(z_1,\bar z_1) \cdots \Phi_{i_s}(z_s,\bar z_s)\rangle \\
& \pm \sum_{l=1}^{s} \xi(z_l)\, \langle \psi_{a_1}(\tau_1) \cdots \psi_{a_m}(\tau_m)\, \Phi_{i_1}(z_1,\bar z_1) \cdots \Phi^{(1,0)}_{i_l}(z_l,\bar z_l) \cdots \Phi_{i_s}(z_s,\bar z_s)\rangle \\
& \pm \sum_{l=1}^{s} \bar\xi(\bar z_l)\, \langle \psi_{a_1}(\tau_1) \cdots \psi_{a_m}(\tau_m)\, \Phi_{i_1}(z_1,\bar z_1) \cdots \Phi^{(0,1)}_{i_l}(z_l,\bar z_l) \cdots \Phi_{i_s}(z_s,\bar z_s)\rangle .
\end{aligned} \tag{4.2}$$

Here, ψ_{a_j} denote the boundary fields at points τ_j on the real line, and Φ_i are the bulk insertions. Furthermore, we have

$$|\psi_a^{(1)}\rangle = G_{-1}|\psi_a\rangle \ , \tag{4.3}$$

with a similar definition for $\Phi^{(1,0)}$ and $\Phi^{(0,1)}$. Finally, the contour integral is taken to surround all fields, and ξ is a globally defined holomorphic vector field on the upper half plane

$$\xi(w) = aw^2 + bw + c \ , \tag{4.4}$$

where a, b, c are real constants. The above signs are uniquely determined by the statistics of the various fields; we shall be more specific momentarily.

We are interested in the special case of a correlator with one bulk insertion Φ, and two boundary insertions ψ_l and ψ_m. The vector field ξ can then be chosen such that the term with an integrated bulk insertion drops out from the Ward identity

$$\xi(w) = (w - z)(w - \bar{z}) \ . \tag{4.5}$$

The Ward identities then give

$$\xi(\tau_l)\ \langle \phi(z)\, \bar{\phi}(\bar{z})\, \psi_l^{(1)}(\tau_l)\, \psi_m(\tau_m)\rangle = \xi(\tau_m)\ \langle \phi(z)\, \bar{\phi}(\bar{z})\, \psi_l(\tau_l)\, \psi_m^{(1)}(\tau_m)\rangle \ . \tag{4.6}$$

Here we have used the usual doubling trick to represent the bulk field $\Phi(z, \bar{z})$ in terms of a chiral field $\phi(z)$ on the upper half plane, together with its image $\bar{\phi}(\bar{z})$ on the lower half plane. We have furthermore used that the statistics of the boundary insertions that are of interest to us is fermionic. One way to see this is in terms of the Landau-Ginzburg theory, where perturbations of the BRST operator $Q \to Q + \lambda\psi$ always involve boundary fermions.

Following [51] we now use the conformal symmetry to evaluate these four-point functions further. For the function on the left hand side we consider the Möbius transformation

$$\hat{f}(u) = \frac{u - \bar{z}}{u - z}\, \frac{\tau_m - z}{\tau_m - \bar{z}} \tag{4.7}$$

which maps the points $z, \bar{z}, \tau_m$ and τ_l to $\infty, 0, 1$ and

$$\hat{\zeta} = \frac{\tau_l - \bar{z}}{\tau_l - z}\, \frac{\tau_m - z}{\tau_m - \bar{z}} \ . \tag{4.8}$$

In the topologically twisted theory ϕ and $\bar{\phi}$ have conformal weight zero, as has ψ_a, while $\psi_a^{(1)}$ has conformal weight one. It thus follows from the usual conformal transformation properties that

$$\begin{aligned}\langle \phi(z)\, \bar{\phi}(\bar{z})\, \psi_l^{(1)}(\tau_l)\, \psi_m(\tau_m)\rangle &= \hat{f}'(\tau_l)\, \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l^{(1)}(\hat{\zeta})\, \psi_m(1)\rangle \\ &= \frac{\bar{z} - z}{(\tau_l - z)^2}\, \frac{\tau_m - z}{\tau_m - \bar{z}}\, \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l^{(1)}(\hat{\zeta})\, \psi_m(1)\rangle \ . \end{aligned} \tag{4.9}$$

Similarly, we use the Möbius transformation

$$f(u) = \frac{u - \bar{z}}{u - z}\, \frac{\tau_l - z}{\tau_l - \bar{z}} \tag{4.10}$$

to rewrite the right hand side as

$$\langle \phi(z)\, \bar{\phi}(\bar{z})\, \psi_l(\tau_l)\, \psi_m^{(1)}(\tau_m) \rangle = \frac{\bar{z}-z}{(\tau_m - z)^2}\, \frac{\tau_l - z}{\tau_l - \bar{z}}\, \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l(1)\, \psi_m^{(1)}(\zeta) \rangle \ , \tag{4.11}$$

where $\zeta = \hat{\zeta}^{-1}$. Using the explicit form of $\xi(\tau_l)$ and $\xi(\tau_m)$ it then follows from the Ward identity (4.6) that

$$\langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l^{(1)}(\hat{\zeta})\, \psi_m(1) \rangle = \zeta^2 \, \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l(1)\, \psi_m^{(1)}(\zeta) \rangle \ . \tag{4.12}$$

The right hand side of (4.1) is now the integral

$$\begin{aligned} \partial_l B_{\Phi\psi_m} &= \int_{C_+} d\hat{\zeta}\ \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l^{(1)}(\hat{\zeta})\, \psi_m(1) \rangle \\ &= \int_{C_+} d\hat{\zeta}\, \hat{\zeta}^{-2}\ \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l(1)\, \psi_m^{(1)}(\zeta) \rangle \\ &= -\int_{C_-} d\zeta\ \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l(1)\, \psi_m^{(1)}(\zeta) \rangle \\ &= +\int_{C_+} d\zeta\ \langle \phi(\infty)\, \bar{\phi}(0)\, \psi_l(1)\, \psi_m^{(1)}(\zeta) \rangle \\ &= \partial_m B_{\Phi\psi_l} \ . \end{aligned} \tag{4.13}$$

Here, the contour C_+ denotes the unit circle with positive orientation and C_- is the circle with opposite orientation. This proves the desired integrability condition.

5. Conclusions

In this paper we have studied the behaviour of a class of D2-branes on the quintic under complex structure deformations of the closed string background. At the Gepner point there are 50 one-dimensional moduli spaces of D2-branes intersecting over certain curves. As one switches on a complex structure deformation of the Gepner point, only a finite number of the corresponding $\mathbb{P}_1$'s remain [1]. This implies that there is an effective space-time superpotential, that possesses a flat direction at the Gepner point, while it only has finitely many discrete minima at generic values of the bulk parameters. From a world-sheet point of view, the interpretation is that the bulk deformation is not truly marginal on world-sheets with boundaries. Rather, it induces a non-trivial RG flow at the boundary, that drives the boundary condition to one that is compatible with the changed background geometry. The relevant driving term can be calculated using matrix factorisation techniques, and the RG flow turns out to be a gradient flow of a potential. This potential can be identified with (a certain contribution to) the effective space-time superpotential that we can calculate explicitly. The calculation is first order in the bulk perturbation, but exact to all orders in the boundary couplings.

We have only studied a certain family of branes on the quintic; it should however be straightforward to generalise our methods to other Calabi-Yau manifolds and other classes of branes. As we have indicated in the main part of the paper one may expect that

the integrated bulk-boundary couplings generalise the familiar bulk periods, and hence the superpotential terms obey a Picard-Fuchs type equation [46, 47, 48]; it would be very interesting to analyse this in detail. Our result should also have an interesting interpretation in the context of mirror symmetry. We have calculated the superpotential for a B-type D-brane on a compact Calabi-Yau manifold which, in particular, is exact to all orders in α'. On the mirror A-side, the superpotential vanishes perturbatively and can only be generated non-perturbatively via disc instantons [52, 53]. Our expression for the B-type superpotential on the D2-branes should therefore have an interpretation in terms of disc instantons on the mirror A-side. One would expect that a Picard-Fuchs like equation would be helpful in analysing this question [48]. We hope to return to these issues in the near future.

A. The cohomology of the factorisations

In this appendix we want to determine the full fermionic cohomology of $U(1)$-charge 1 for the factorisations $Q(a,b,c)$ (2.8) with $a \neq 0$. First we observe that the coordinates involved in J_1 and E_1 (namely x_1 and x_2) do not appear in J_4, E_4 or J_5, E_5. Therefore the cohomology H of Q separates into

$$H(Q) = H(Q_1) \odot H(Q_2) \ , \tag{A.1}$$

where Q_1 and Q_2 are the separate factorisations

$$\begin{aligned} Q_1 &= \pi^1 J_1 + \bar{\pi}^1 E_1 \ , \\ Q_2 &= \pi^4 J_4 + \pi^5 J_5 + \bar{\pi}^4 E_4 + \bar{\pi}^5 E_5 \ . \end{aligned} \tag{A.2}$$

The explicit polynomials are

$$\begin{aligned} J_1 &= x_1 - \eta x_2 & E_1 &= \prod_{\eta'^5=-1,\eta'\neq\eta} (x_1 - \eta' x_2) \\ J_4 &= a x_4 - b x_3 & E_4 &= \frac{1}{a^5}\left(b^4 x_3^4 + a b^3 x_3^3 x_4 + a^2 b^2 x_3^2 x_4^2 + a^3 b x_3 x_4^3 + a^4 x_4^4\right) \\ J_5 &= c x_3 - a x_5 & E_5 &= -\frac{1}{a^5}\left(c^4 x_3^4 + a c^3 x_3^3 x_5 + a^2 c^2 x_3^2 x_5^2 + a^3 c x_3 x_5^3 + a^4 x_5^4\right) \ . \end{aligned} \tag{A.3}$$

The cohomology of Q_1 has been calculated in [41, 39, 40], and consists of four bosonic elements of $U(1)$-charge 0, $\frac{2}{5}$, $\frac{4}{5}$ and $\frac{6}{5}$, respectively; it does not contain any fermions at all. Thus in order to obtain a fermion of the full factorisation, we need to tensor one of these bosons with a fermion from Q_2. We are only interested in fermions of Q of total $U(1)$-charge 1. Since the $U(1)$-charge of the fermions in Q_2 is always positive, there are three cases to consider: the fermions in the cohomology of Q_2 can have $U(1)$-charges 1, $\frac{3}{5}$ or $\frac{1}{5}$ which together with the boson of Q_1 of $U(1)$-charges 0, $\frac{2}{5}$ or $\frac{4}{5}$, respectively, then produce a fermion of total $U(1)$-charge 1. Thus it is sufficient to analyse the fermionic cohomology of Q_2 for these three $U(1)$-charges separately.

A.1 The Q_2-fermions of charge 1

The general Q_2-closed fermion has an expansion (the closure conditions force the absence of any higher powers of boundary fermions)

$$\psi = \pi^4 p_4 + \bar{\pi}^4 m_4 + \pi^5 p_5 + \bar{\pi}^5 m_5 \ , \tag{A.4}$$

where we have dropped some exact terms — see (A.7) below. The requirement that ψ has $U(1)$-charge 1 implies that p_4 and p_5 are polynomials of degree 1 (thus each p_i has 3 parameters) while m_4 and m_5 are polynomials of degree 4 (with 15 parameters each), giving in total 36 parameters. The condition that ψ is closed implies further that

$$J_4\, m_4 + J_5\, m_5 + E_4\, p_4 + E_5\, p_5 = 0 \ . \tag{A.5}$$

The left hand side is a homogeneous polynomial of degree 5, and hence represents 21 conditions. We have checked (using standard matrix techniques) that these 21 conditions are independent. This implies that the space of closed fermions of the $U(1)$-charge 1 is 15-dimensional.

It remains to determine how many of them are exact. To see this we make the following ansatz for the most general boson,

$$\Lambda = \hat{a} + \hat{b}\pi^4\bar{\pi}^4 + \hat{c}\pi^4\pi^5 + \hat{d}\pi^4\bar{\pi}^5 + \hat{e}\bar{\pi}^4\pi^5 + \hat{f}\bar{\pi}^4\bar{\pi}^5 + \hat{g}\pi^5\bar{\pi}^5 + \hat{h}\pi^4\bar{\pi}^4\pi^5\bar{\pi}^5 \ . \tag{A.6}$$

Then

$$\begin{aligned}
[Q,\Lambda] =& \pi^4\left(-\hat{b}J_4 - \hat{d}J_5 - \hat{c}E_5\right) + \pi^5\left(\hat{e}J_4 - \hat{g}J_5 + \hat{c}E_4\right) \\
&+ \bar{\pi}^4\left(\hat{b}E_4 - \hat{e}E_5 - \hat{f}J_5\right) + \bar{\pi}^4\left(\hat{d}E_4 + \hat{g}E_5 + \hat{e}J_4\right) \\
&- \pi^4\bar{\pi}^4\pi^5\hat{h}J_5 + \pi^4\bar{\pi}^4\bar{\pi}^5\hat{h}E_5 - \pi^4\pi^5\bar{\pi}^5\hat{h}J_4 + \bar{\pi}^4\pi^5\bar{\pi}^5\hat{h}E_4 \ .
\end{aligned} \tag{A.7}$$

Consistency with the ansatz for ψ requires $\hat{h} = 0$ and $\hat{c} = 0$. Moreover $\hat{a}$ can be set to zero, too. The other parameters must be polynomials of degree 0, except for $\hat{f}$ which has to have degree 3 (and therefore 10 parameters). In total the space of exact fermionis is described by 14 parameters. Again, using standard matrix methods, we have shown that these 14 parameters are linearly independent. This implies that the fermionic cohomology of Q_2 of $U(1)$-charge 1 is 1-dimensional. A representative of the corresponding cohomology class for Q is (for $c \neq 0$)

$$\psi_1 = \partial_b Q \tag{A.8}$$

or explicitly

$$\psi_1 = -\, x_3\pi^4 + \frac{1}{a^5}\left[4b^3x_3^4 + 3ab^2x_3^3x_4 + 2a^2bx_3^2x_4^2 + a^3x_3x_4^3\right]\bar{\pi}^4 \tag{A.9}$$

$$-\,\frac{b^4}{c^4}x_3\pi^5 + \frac{b^4}{a^5c^4}\left[4c^3x_3^4 + 3ac^2x_3^3x_5 + 2a^2cx_3^2x_5^2 + a^3x_3x_5^3\right]\bar{\pi}^5 \ . \tag{A.10}$$

A.2 The Q_2-fermions of charge $\frac{3}{5}$

The same arguments can be used to determine the fermions of $U(1)$-charge $\frac{3}{5}$. In this case, p_4 and p_5 have both degree 0 (*i.e.* are constants) while m_4 and m_5 have both degree 3 (with 10 parameters each), giving rise to 22 parameters. The closure condition is now given by a polynomial of degree 4, leading to 15 (independent) equations. Thus the space of closed fermions is in this case 7-dimensional.

For exact fermions we find that they are described by bosons Λ with $\hat{a} = 0$, $\hat{b} = 0$, $\hat{d} = 0$, $\hat{c} = 0$, $\hat{e} = 0$, $\hat{g} = 0$, $\hat{h} = 0$ and $\hat{f}$ a polynomial of degree 2 (with 6 parameters). Thus there are 6 different exact fermions, and we have checked that they are in fact linearly independent. This implies that there is precisely one fermion of charge $\frac{3}{5}$ in the cohomology of Q_2. A representative of the corresponding cohomology class for Q is given by (for $c \neq 0$)

$$\begin{aligned}\psi_2 = x_1\, \partial_b \Big[& b\pi^4 - c\pi^5 - (b^4 x_3^3 + b^3 x_3^2 x_4 + b^2 x_3 x_4^2 + b x_4^3)\bar{\pi}^4 \\ & + (c^4 x_3^3 + c^3 x_3^2 x_5 + c^2 x_3 x_5^2 + c x_5^3)\bar{\pi}^5 \Big] \,, \end{aligned} \tag{A.11}$$

or, since ψ_1 is proportional to x_3,

$$\psi_2 = \frac{x_1}{x_3}\psi_1 \ . \tag{A.12}$$

A.3 The Q_2-fermions of charge $\frac{1}{5}$

For fermions of charge $\frac{1}{5}$, our ansatz has 12 parameters, and the closure condition leads to 9 linearly independent conditions. Thus there are 3 different closed fermions. In Λ, all parameters are zero except $\hat{f}$, which is a polynomial of degree 1 with 3 independent parameters. This implies that all 3 closed fermions are in fact exact, and hence that the cohomology is trivial.

B. Calculating the superpotential

In this appendix we give details about how to calculate the effective superpotential $\mathcal{W}$ explicitly. We begin by recalling some facts about differentials on Fermat curves.

B.1 Differentials on the Fermat curve and their integrals

Let us consider the Fermat curve defined by

$$\hat{b}^5 + \hat{c}^5 = 1 \ . \tag{B.1}$$

For $a = 1$ this is the curve that describes the brane moduli space $1 + b^5 + c^5 = 0$ provided we identify $\hat{b} = -b$ and $\hat{c} = -c$. The general theory of globally defined differentials is described in [54]. The simplest class of differentials, the differentials of the first kind, are those that are holomorphic on the full curve. They are of the form

$$\omega_{rs} = \hat{b}^r \hat{c}^s\, \frac{\frac{1}{5} d(\hat{b}^5)}{\hat{b}^5 \hat{c}^5} = \hat{b}^{r-1}\hat{c}^{s-1}\frac{d\hat{b}}{\hat{c}^4} \ , \tag{B.2}$$

where $r, s, \geq 1$. Since $\hat{b}^4 d\hat{b} = -\hat{c}^4 d\hat{c}$ this is equivalent to

$$\omega_{rs} = -\hat{b}^{r-1}\hat{c}^{s-1}\frac{d\hat{c}}{\hat{b}^4} \ . \tag{B.3}$$

The first formula (B.2) is defined on the patch of the moduli space where $\hat{c} \neq 0$, while the second (B.3) is defined for $\hat{b} \neq 0$. Since on (B.1) $\hat{b}\hat{c} \neq 0$ at least one of these two expressions is everywhere well-defined. In particular, this therefore proves that the differentials $\omega_{r,s}$ are holomorphic for finite $\hat{b}$ and $\hat{c}$. The only potential poles may thus appear at $\hat{b}, \hat{c} = \infty$. Expanding around $\hat{b} = \infty$ shows that the differentials are finite as long as $r + s \leq 4$. Therefore we find the holomorphic differentials (for $\hat{c} \neq 0$)

$$\frac{1}{\hat{c}^4}d\hat{b},\ \frac{1}{\hat{c}^3}d\hat{b},\ \frac{1}{\hat{c}^2}d\hat{b},\ \frac{b}{\hat{c}^4}d\hat{b},\ \frac{\hat{b}^2}{\hat{c}^4}d\hat{b},\ \frac{\hat{b}}{\hat{c}^3}d\hat{b} \ . \tag{B.4}$$

In fact this is a basis for the holomorphic differentials on the curve. Its number is equal to the genus of the curve.

B.1.1 Integrating the holomorphic differentials

In order to calculate the effective superpotential we need to integrate these holomorphic differentials. For all of them the answer can be expressed in terms of a hypergeometric function. In fact in the chart where $\hat{c} \neq 0$ we have

$$\int_0^{\hat{b}} \omega_{rs} = \int_0^{\hat{b}} d\tilde{b}\, \frac{\tilde{b}^{r-1}\hat{c}(\tilde{b})^{s-1}}{\hat{c}(\tilde{b})^4} = \frac{1}{r}\,\hat{b}^r\ {}_2\mathrm{F}_1(\tfrac{r}{5}, 1-\tfrac{s}{5}; 1+\tfrac{r}{5}; \hat{b}^5) \ . \tag{B.5}$$

On the other hand in the chart with $\hat{b} \neq 0$ we get instead

$$\int_0^{\hat{c}} \omega_{rs} = -\int_0^{\hat{c}} d\tilde{c}\, \frac{\hat{b}(\tilde{c})^{r-1}\tilde{c}^{s-1}}{\hat{b}(\tilde{c})^4} = -\frac{1}{s}\hat{c}^s\ {}_2\mathrm{F}_1(\tfrac{s}{5}, 1-\tfrac{r}{5}; 1+\tfrac{s}{5}; \hat{c}^5) \ . \tag{B.6}$$

In particular, the formula for the effective superpotential (3.14) follows directly from (B.5). Note that the reference point $\hat{b}_0 = 0$ corresponds to $\hat{c}_0^5 = 1$, and vice versa.

B.1.2 Comparing different charts

Since the differentials we have integrated are globally defined, the two expressions we obtain in different charts, namely (B.5) and (B.6), must agree, once we have taken into account that the lower bound of the integrals are different. This can also be checked explicitly. In order to see this we use the identity

$$\begin{aligned} {}_2\mathrm{F}_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; 1-z) = {} & \frac{\Gamma(\mathfrak{c})\Gamma(\mathfrak{a}+\mathfrak{b}-\mathfrak{c})}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{b})}{}_2\mathrm{F}_1(\mathfrak{c}-\mathfrak{a}, \mathfrak{c}-\mathfrak{b}; \mathfrak{c}-\mathfrak{a}-\mathfrak{b}+1; z)\, z^{\mathfrak{c}-\mathfrak{a}-\mathfrak{b}} \\ & + \frac{\Gamma(\mathfrak{c})\Gamma(\mathfrak{c}-\mathfrak{a}-\mathfrak{b})}{\Gamma(\mathfrak{c}-\mathfrak{a})\Gamma(\mathfrak{c}-\mathfrak{b})}{}_2\mathrm{F}_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{a}+\mathfrak{b}-\mathfrak{c}+1; z) \ . \end{aligned} \tag{B.7}$$

This allows us to rewrite the right hand side of (B.5) as

$$\frac{1}{r}\hat{b}^r\hat{c}^s\,\frac{\Gamma(1+\frac{r}{5})\Gamma(-\frac{s}{5})}{\Gamma(\frac{r}{5})\Gamma(1-\frac{s}{5})}\ {}_2\mathrm{F}_1(1, \tfrac{r+s}{5}; 1+\tfrac{s}{5}; \hat{c}^5) + \frac{1}{r}\hat{b}^r\ \frac{\Gamma(1+\frac{r}{5})\Gamma(\frac{s}{5})}{\Gamma(\frac{r+s}{5})}\ {}_2\mathrm{F}_1(\tfrac{r}{5}; 1-\tfrac{s}{5}; 1-\tfrac{s}{5}; \hat{c}^5) \ . \tag{B.8}$$

With the help of the identities

$$
{}_2\mathrm{F}_1(\mathfrak{a}, \mathfrak{c}; \mathfrak{c}; z) = (1-z)^{-\mathfrak{a}} \tag{B.9}
$$

$$
{}_2\mathrm{F}_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; z) = (1-z)^{\mathfrak{c}-\mathfrak{a}-\mathfrak{b}}\, {}_2\mathrm{F}_1(\mathfrak{c}-\mathfrak{a}, \mathfrak{c}-\mathfrak{b}; \mathfrak{c}; z) \tag{B.10}
$$

as well as properties of the Γ-function, (B.8) then becomes

$$
-\frac{1}{s}\hat{c}^s\, {}_2\mathrm{F}_1(\tfrac{s}{5}, 1-\tfrac{r}{5}; 1+\tfrac{s}{5}; \hat{c}^5) + \frac{1}{r}\frac{\Gamma(1+\frac{r}{5})\Gamma(\frac{s}{5})}{\Gamma(\frac{r+s}{5})} \ . \tag{B.11}
$$

By the Gauss hypergeometric theorem the second term is precisely the value of the right hand side of (B.5) for $\hat{b} = 1$, while the first term agrees with (B.6). Since $\hat{b} = 1$ corresponds to $\hat{c} = 0$, the second term just accounts for the fact that the reference points in the two line integrals (B.5) and (B.6) are different, and we have therefore proven our claim. In particular, this then implies that the function $\mathcal{W}$ defined by (3.14) solves both (3.11) and (3.12).

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